

FUNTZIOAK

→ Bi aldagaien arteko erlazioa

4 ERATAN ADIERAZI

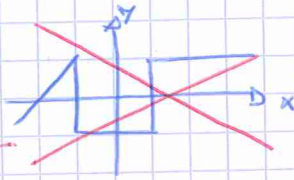
- Adierazpen aljebraikoa
- Inguirugilea
- Balio taulen bidez
- Adierazpen grafikoa

aldagai askea: x
menpeko aldagaia: y

non x bakoitarrak y bakoitiak dagoen

Ad: $y = \sqrt{x}$

x	2	3	6	2	1
y	5	4	2	7	8



Ez da funtzioa x bakoiti y bat bakoiti gertatzen dago

FUNTZIO FAMILIA

LINEALA

$y = mx + n$
malda = $\frac{\Delta y}{\Delta x}$

ordenatzen ordenatua

x	x_1	x_2	x_3	$x_4 \dots$
y	y_1	y_2	y_3	$y_4 \dots$

$$\frac{y_1 - y_3}{x_4 - x_1} = "$$

INJEKTIBOA

x bati y bat
 y bati x bat } ezagutari nagusia

FUNTZIO IRRAZIONALAK

$$y = \sqrt{(x^3 - 2x)} 3x$$

$$E_r(f) = \mathbb{R} \geq 0$$

$$E_{\text{irrazional}}(f) = 0(2), \sqrt{2}, -\sqrt{2}$$

$$\begin{array}{c} x(x^2 - 2) 3x \\ \downarrow \quad \downarrow \quad \downarrow \\ 0 \quad \sqrt{2} \quad 0 \\ -\sqrt{2} \end{array}$$



$$E_r(f) = (-\infty, -\sqrt{2}] \cup [0, \sqrt{2}] \cup [\sqrt{2}, \infty)$$

→ Funtzio baten eremu, aldagai askea hartzen duen $E_r(f)$

→ Irilkerak edo irratia mendeko aldagaiak hartzen duen $I_b(f)$ edo $I_r(f)$

FUNTZIOEN ARTEKO ERAGIKETAK

$$\begin{aligned} f(x) &= x+3 \\ g(x) &= x^2-4 \end{aligned} \quad \left\{ \begin{aligned} f(x)+g(x) &= z(x) \\ x+3+x^2-4 &= z(x) \\ z(x) &= x^2+x-1 \end{aligned} \right. \quad \text{Er}(z) = \mathbb{R}$$

$$\begin{aligned} f(x)-g(x) &= h(x) \\ x+3-(x^2-4) &= h(x) \\ x+3-x^2+4 &= h(x) \\ -x^2+x+7 &= h(x) \end{aligned} \quad \text{Er}(h) = \mathbb{R}$$

$$\begin{aligned} f(x) \cdot g(x) &= m(x) \\ (x+3)(x^2-4) &= m(x) \\ x^3-4x+3x^2-12 &= m(x) \\ x^3+3x^2-4x-12 &= m(x) \end{aligned} \quad \text{Er}(m) = \mathbb{R}$$

$$\frac{f(x)}{g(x)} = n(x)$$

$$\begin{aligned} \frac{x+3}{x^2-4} &= n(x) & \text{Er}(n) &= \mathbb{R} - \{2, -2\} \\ \hookrightarrow x^2-4 &= 0 \\ x^2 &= 4 \\ (-2)^2 &= 4 \\ 2^2 &= 4 \end{aligned}$$

FUNTZIOEN KONPOSIZIOA

$$\begin{aligned} f(x) &= 2x \\ g(x) &= x^2-1 \end{aligned} \quad \left\{ \begin{aligned} (f \circ g)(x) &= f(g(x)) & f(x^2-1) &= 2(x^2-1) \\ (g \circ f)(x) &= g(f(x)) & f(x^2-1) &= 2x-2 = f(g(x)) \end{aligned} \right.$$

ALDERANTZIKO FUNTZIOA

Alderantzizko funtzioa egon ahal izateko funtzioa injektiboa izan behar da.

$$\text{Er}(f) = \text{Ib}(f^{-1})$$

$$\text{Er}(f^{-1}) = \text{Ib}(f)$$

PAUSOAK

- 1- $y = f(x)$ berdinetan x bakarku
- 2- aldagaia izatea trukat

} LD

PAUSOAK

- 1- x eta y trukat
- 2- y bakarku

} Amaia

$$\begin{aligned} y &= x^3+3 \\ 1 \quad x &= y^3+3 \\ 2 \quad x-3 &= y^3 \\ \sqrt[3]{x-3} &= y \\ y &= \sqrt[3]{x-3} \end{aligned}$$

$$\text{Er}(y) = \mathbb{R}$$

$$\begin{aligned} y &= x^3+3 \\ y-3 &= x^3 \\ y-3 &= \sqrt[3]{x-3} \end{aligned} \quad \left\{ \begin{aligned} & \text{Alderantzizkoa} \end{aligned} \right.$$

ARDATZEKIKO EBAKIDURA PUNTUA

$$f(x) = x^2 - 5x + 6$$

$$f(0) = 6$$

$$EP \begin{cases} ox \rightarrow (3, 0); (2, 0) \\ oy \rightarrow (0, 6) \end{cases}$$

$$0 = x^2 - 5x + 6$$

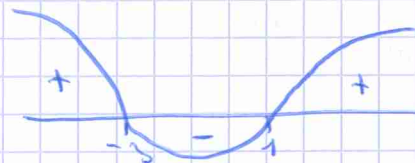
$$\frac{5 \pm \sqrt{5^2 - 4 \cdot 1 \cdot 6}}{2 \cdot 1} = \frac{5 \pm 1}{2} \rightarrow \begin{matrix} 3 \\ 2 \end{matrix}$$

FUNTZIO BATEK ZERNUA

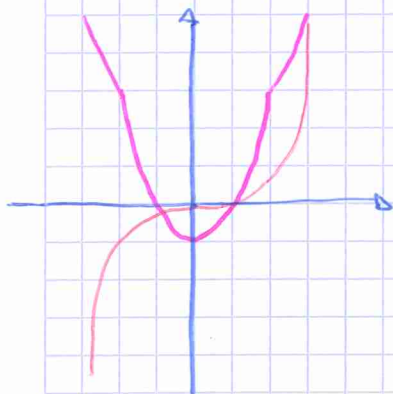
$$p(x) = x^2 + 2x - 3$$

$$0 = x^2 + 2x - 3$$

$$\frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-3)}}{2 \cdot 1} = \frac{-2 \pm 4}{2} \rightarrow \begin{matrix} 1 \\ -3 \end{matrix}$$



SIMETRIA



Bikotia = y-erako simetria $f(x) = f(-x)$
 Bokotia = x-erako simetria $f(-x) = -f(x)$

$$f(x) = x^2 + 3$$

$$f(-x) = (-x)^2 + 3$$

$$f(-x) = x^2 + 3$$

$$f(x) = 3x - 5$$

$$f(-x) = 3(-x) - 5$$

$$f(-x) = -3x - 5$$

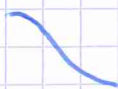
$$-f(-x) = 3x + 5 \neq f(x)$$

PERIODIKotasuna

$$f(x) = f(x + T)$$

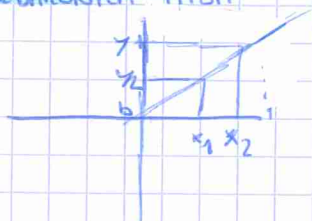
$$f(225) = f(225 - 32.7)$$

JARRAITUTASUNA



→ Hainbaiek
adibidea

ALDAUNTZA TASA



$$y = ax + b$$

$$a = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_2}{x_1 - x_2}$$

↳ malda

MALDA
KONST

$$BBAT(a, b) = \frac{f(b) - f(a)}{b - a}$$

MALDA ET
KONST

BORNEAK ETA ASINTOTAK

NOLA KALKULATU ASINTOTAK

1- $f(x) = \frac{A(x)}{B(x)}$ analitikoki

VERTIKALA Ekuazioa $x = a$ non $a \in \mathbb{R}$ den

- 1 - Izendatzailea kalkulatu, 0 bikoizten duen x balore errealak. Horiek dira posible diren asintota bertikalak.
- 2 - Bepatu ea balore horiek et diren zenbakitzailearen erroak. Et bada, asintota bertikala dira, bestela ez.

Adib: $\frac{3x^2 - 1}{x^2 - 1} \rightarrow 2 \text{ asintota } \rightarrow x = -1$
 $\rightarrow x = 1$

$\begin{matrix} \uparrow \\ -1 & 1 \end{matrix}$

↳ gero $3x^2 - 1$ -ari oinarritu eta et duen x balore behar

HORIZONTALA Ekuazioa $y = b$ non $b \in \mathbb{R}$ den

- 1 - Horizontala izateko funtzio aritmetikoaren zenbakitzailearen maila izendatzailearen maila $\leq$ edo $<$ (txikiagoa) izan behar du.
- 2 - Aritmetiko funtzioaren maila altuena ikusi gero maila hori duen koef. zenbakitzailearen zati maila horien duen koef. izendatzailearen. Eta emaitza hura b da.

ZENITARRA Ekuazioa $y = ax + b$ non $a \in \mathbb{R}$ den

- 1 - zenbakitzailearen maila izendatzailearen maila unitate baten gainetik
- 2 -

Adib:

1- $\frac{3x^2}{2x - 1} \rightarrow n = 2$
 $2x - 1 \rightarrow n = 1$

01NARRIZKO FUNTZIOAK

- NOTAK**
- Funtzio lineala (1)
 - Funtzio koadratikoa (2)
 - Funtzio esponentziala
 - Funtzio logaritmikoa
 - Aldeantzerako proportzionaltasuneko funtzioa
 - Funtzio trigonometrikoa

(1) $y = ax + b$ → jatorri gideratua $f(0) = -$
 ↓ malda
 $a = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$

(2) $6x^2 + 2y = 1/4$
 $6x^2 - \frac{1}{4} = -2y$

$\frac{6x^2 - \frac{1}{4}}{-2} = y$

$y = -3x^2 + \frac{1}{8}$

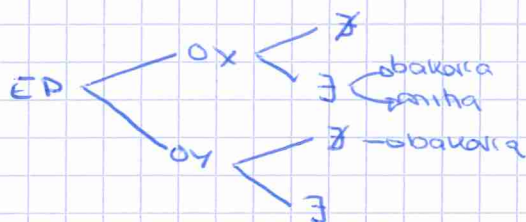
$f(x) = ax^2 + bx + c$

↳ Simetria ardatzaren formula

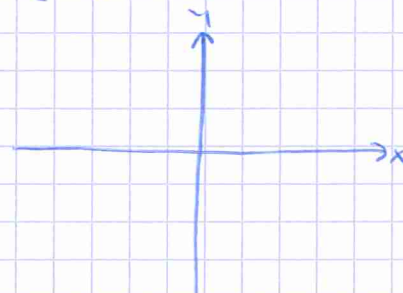
$x = \frac{-b}{2a} = x = 0$

Max/min → $(0, \frac{1}{8})$

EP $\begin{cases} OX \\ OY \end{cases} (0, f(0))$



$a > 0$ U
 $a < 0$ ∩



FUNTZIO ABRAZIONALAK

$y = \frac{A(x)}{B(x)}$ $B(x)$ -ren maila ≥ 1

• Zeraugarri: mugurak $E_r(f) = D_R - \{x \in R / Q(x) = 0\}$

• Zeruua

• EP $\begin{cases} OX (f^{-1}(0), 0) \\ OY (0, f(0)) \end{cases}$

• Asintotak

ALDERANTZAKO PROPORTZIONALITASUNEKO FUNTZIOAK

$$y = \frac{k}{x} \quad x \cdot y = k$$

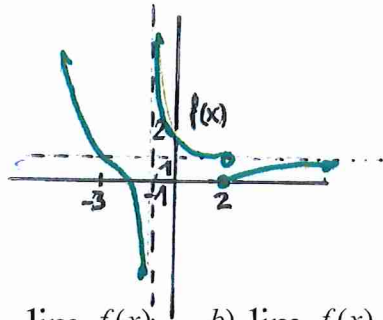
FUNTZIO LOG

$$y = a^x \quad \text{alderantzakoa}$$

$$x = a^y \rightarrow y = \log_a x = f^{-1}(x)$$

LIMITEEN KALKULUA. (TEST).

1) $f(x)$ grafiittik abiatute kalkulatun ondoengo limiteak:



- a) $\lim_{x \rightarrow -\infty} f(x)$ b) $\lim_{x \rightarrow +\infty} f(x)$ c) $\lim_{x \rightarrow -3} f(x)$ d) $\lim_{x \rightarrow 0} f(x)$
 e) $\lim_{x \rightarrow 2^-} f(x)$ f) $\lim_{x \rightarrow 2^+} f(x)$ g) $\lim_{x \rightarrow 2} f(x)$ h) $\lim_{x \rightarrow -1} f(x)$

2) Kalkulatun ondoengo limiteak

a) $\lim_{x \rightarrow 3} \frac{x^3 - 4x^2 - 3x + 18}{x^3 + 3x^2 - 9x - 27}$

b) $\lim_{x \rightarrow \infty} (x-1) \cdot \frac{2x}{x^2 + 4}$

c) $\lim_{x \rightarrow 1} \left(\frac{x+1}{x-1} - \frac{x^2+2}{x^2-1} \right)$

d) $\lim_{x \rightarrow 0} (1+2x)^{\frac{2+x}{x}}$

e) $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 2x + 7} - x)$

f) $\lim_{x \rightarrow 2} \frac{x-2}{\sqrt{x+2} - 2}$

g) $\lim_{x \rightarrow 2} \frac{-x^2 + 2x - 1}{x^4 - 6x^3 + 13x^2 - 12x + 4}$

h) $\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 2x}}{\sqrt{4x^2 + 5x}}$

$$5) \lim_{x \rightarrow 0} (1+2x)^{\frac{2+x}{x}} \stackrel{\infty/\infty \text{ ind.}}{=} e^{\lim_{x \rightarrow 0} \frac{2+x}{x} (1+2x-1)} \stackrel{\lim_{x \rightarrow 0} \frac{(2+x) \cdot 2x}{x}}{=} e^4 = \boxed{e^4} \quad 1$$

$$6) \lim_{x \rightarrow \infty} (\sqrt{x^2+2x+7} - x) \stackrel{\infty-\infty \text{ ind.}}{=} \lim_{x \rightarrow \infty} \frac{(\sqrt{x^2+2x+7} - x)(\sqrt{x^2+2x+7} + x)}{(\sqrt{x^2+2x+7} + x)} =$$

$$= \lim_{x \rightarrow \infty} \frac{x^2+2x+7-x^2}{\sqrt{x^2+2x+7} + x} \stackrel{\infty/\infty \text{ ind.}}{=} \frac{2}{1+1} = \boxed{1} \quad 1$$

$$7) \lim_{x \rightarrow 2} \frac{x-2}{\sqrt{x+2}-2} \stackrel{0/0 \text{ ind.}}{=} \lim_{x \rightarrow 2} \frac{(x-2)(\sqrt{x+2}+2)}{(\sqrt{x+2}-2)(\sqrt{x+2}+2)} = \lim_{x \rightarrow 2} \frac{(x-2)(\sqrt{x+2}+2)}{(x-2)} = \boxed{4} \quad 1$$

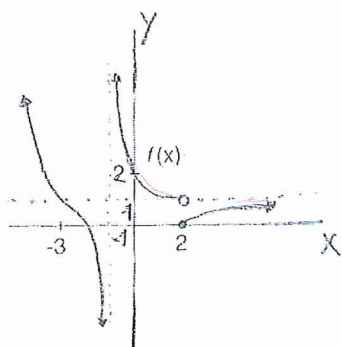
$$8) \lim_{x \rightarrow 2} \frac{-x^2+2x-1}{x^4-6x^3+13x^2-12x+4} \stackrel{-1/0 \text{ ind. } \frac{\infty}{0}}{=} \boxed{-\infty} \quad 1$$

$$\begin{cases} \lim_{x \rightarrow 2^-} \frac{-x^2+2x-1}{x^4-6x^3+13x^2-12x+4} = -\infty \\ \lim_{x \rightarrow 2^+} \frac{-x^2+2x-1}{x^4-6x^3+13x^2-12x+4} = -\infty \end{cases}$$

$$9) \lim_{x \rightarrow \infty} \frac{\sqrt{x^2+2x}}{\sqrt{4x^2+5x}} \stackrel{\infty/\infty \text{ ind.}}{=} \sqrt{\frac{1}{4}} = \boxed{\frac{1}{2}} \quad 1$$

ZUZENKETA TEST.

1)



2p

$$a) \lim_{x \rightarrow -\infty} f(x) = \infty \quad 0^2$$

$$b) \lim_{x \rightarrow \infty} f(x) = 1 \quad 0^2$$

$$c) \lim_{x \rightarrow -3} f(x) = 1 \quad 0^3 \quad \begin{cases} \lim_{x \rightarrow -3^+} f(x) = 1 \\ \lim_{x \rightarrow -3^-} f(x) = 1 \end{cases}$$

$$d) \lim_{x \rightarrow 0} f(x) = 2 \quad 0^3 \quad \begin{cases} \lim_{x \rightarrow 0^-} f(x) = 2 \\ \lim_{x \rightarrow 0^+} f(x) = 2 \end{cases}$$

$$e) \lim_{x \rightarrow 2^-} f(x) = 1 \quad 0^2$$

$$f) \lim_{x \rightarrow 2^+} f(x) = 0 \quad 0^2$$

$$g) \lim_{x \rightarrow 2} f(x) = \text{not defined} \quad 0^2$$

$$h) \lim_{x \rightarrow -1} f(x) = \text{not defined} \quad 0^3$$

$$\begin{cases} \lim_{x \rightarrow -1^-} f(x) = -\infty \\ \lim_{x \rightarrow -1^+} f(x) = \infty \end{cases}$$

$$2) \lim_{x \rightarrow 3} \frac{x^3 - 4x^2 - 3x + 18}{x^3 + 3x^2 - 9x - 27} = \frac{0}{0} \text{ ind.} = \lim_{x \rightarrow 3} \frac{(x-3)(x-3)(x+2)}{(x-3)(x^2+6x+9)} = \frac{0}{36} = 0 \quad 1$$

$$x^3 - 4x^2 - 3x + 18 = (x-3)(x-3)(x+2)$$

$$\begin{array}{r|rrrr} 1 & 1 & -4 & -3 & 18 \\ 3 & & 3 & -3 & -18 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

$$x^3 + 3x^2 - 9x - 27 =$$

$$\begin{array}{r|rrrr} 1 & 1 & 3 & -9 & -27 \\ 3 & & 3 & 18 & 27 \\ \hline & 1 & 6 & 9 & 0 \end{array}$$

$$3) \lim_{x \rightarrow \infty} (x-1) \cdot \frac{2x}{x^2+4} = 2 \quad 1$$

$$4) \lim_{x \rightarrow 1} \left(\frac{x+1}{x-1} - \frac{x^2+2}{x^2-1} \right) = \frac{\infty - \infty}{\infty} \text{ ind.} = \lim_{x \rightarrow 1} \frac{x+1+2x-x^2-2}{(x-1)(x+1)} = \lim_{x \rightarrow 1} \frac{2x-1}{x^2-1} = \text{not defined} \quad 1$$

$$\begin{cases} \lim_{x \rightarrow 1^-} \frac{2x-1}{x^2-1} = -\infty \\ \lim_{x \rightarrow 1^+} \frac{2x-1}{x^2-1} = \infty \end{cases}$$

$$\frac{k}{0}$$

$x \rightarrow \pm\infty$ denoan limitea $\frac{k}{0}$ ematen badigu ez da indeterminazioa $(+\infty, \text{edo}, -\infty)$ bakarrik alde batetik gertatzen da. Kasu honetan $\frac{(+)}{(+)} \rightarrow +\infty$ edo $\frac{(-)}{(-)} \rightarrow +\infty$ edo $\frac{(+)}{(-)} \rightarrow -\infty$ edo $\frac{(-)}{(+)} \rightarrow -\infty$.

Ad. $\lim_{x \rightarrow \infty} \frac{6x^4+1}{2x^3+1} = \frac{\infty}{\infty} \rightarrow \frac{6}{2} = 3$

Beraz $\frac{k}{0}$ indeterminazioa kontsideratuko dugu $x \rightarrow a$ agertzen denoan kasu honetan

$$\left[\frac{k}{0} \right] < \begin{matrix} -\infty, \infty \text{ edo } \exists \\ \infty, -\infty \text{ edo } \exists \end{matrix} \left. \vphantom{\begin{matrix} -\infty, \infty \text{ edo } \exists \\ \infty, -\infty \text{ edo } \exists \end{matrix}} \right\} \begin{matrix} \text{Albo limiteak kalkulat} \\ \text{biak bat badotz} \\ \text{limitea } \exists \text{ da eta ez} \\ \text{badotz bat } \exists \end{matrix}$$

Ad:

$\lim_{x \rightarrow \frac{1}{2}} \frac{x^2-3x+1}{2x-1} = \frac{(\frac{1}{2})^2-3(\frac{1}{2})+1}{2(\frac{1}{2})-1} = \frac{-1/4}{0} \rightarrow \text{ind } \frac{k}{0} \rightarrow \frac{k < 0}{0} \rightarrow -\infty$

$$\left\{ \begin{matrix} \lim_{x \rightarrow (\frac{1}{2})^-} \frac{x^2-3x+1}{2x-1} = \infty \\ \lim_{x \rightarrow (\frac{1}{2})^+} \frac{x^2-3x+1}{2x-1} = -\infty \end{matrix} \right\} \Rightarrow \lim_{x \rightarrow \frac{1}{2}} \frac{x^2-3x+1}{2x-1} = \exists$$

$\lim_{x \rightarrow 0} \frac{4}{x^2} = \frac{4}{0} \text{ ind } \frac{k}{0} \rightarrow \frac{4}{0} \rightarrow \infty$

$$\left\{ \begin{matrix} \lim_{x \rightarrow 0^-} \frac{4}{x^2} = \infty \\ \lim_{x \rightarrow 0^+} \frac{4}{x^2} = \infty \end{matrix} \right\} = \infty$$

$\lim_{x \rightarrow 5} \frac{x^2+5x-14}{x^2-10x+25} = \frac{36}{0} \text{ ind } \frac{k}{0} \rightarrow \frac{36}{0} \rightarrow \infty$

bi aukerak $\infty - k > 0 \rightarrow \exists$

$\lim_{x \rightarrow 2} \frac{-x^2+2x-1}{x^4-6x^3+13x^2-12x+4} = \frac{-1}{0} \text{ ind } \frac{k}{0} \rightarrow \frac{-1}{0} \rightarrow -\infty$

$$\left\{ \begin{matrix} \lim_{x \rightarrow 2^-} \frac{-x^2+2x-1}{x^4-6x^3+13x^2-12x+4} = -\infty \\ \lim_{x \rightarrow 2^+} \frac{-x^2+2x-1}{x^4-6x^3+13x^2-12x+4} = -\infty \end{matrix} \right\} = -\infty$$

③

$$\lim_{\substack{x \rightarrow \infty \\ x \rightarrow -\infty}} \frac{P(x)}{Q(x)} = \frac{\infty}{\infty} \text{ indet.}$$

Indeterminazio ebatzeko 3 aukera desberdin.

① $P(x)$ -ren maila = $Q(x)$ -ren maila $\Rightarrow \lim_{\substack{x \rightarrow \infty \\ x \rightarrow -\infty}} \frac{P(x)}{Q(x)} = \boxed{K}$

$$K = \frac{P(x)\text{-ren maila altuen koefizientea}}{Q(x)\text{-ren maila altuen koefizientea}}$$

Ad. $\lim_{x \rightarrow \infty} \frac{3x^3 + 2x^2 - 5}{5x + 9x^3} = \frac{\infty}{\infty} \text{ ind.} = \frac{3}{9} = \boxed{\frac{1}{3}}$

$P(x)$ -ren maila 3.

$Q(x)$ -ren maila 3.

Ad. $\lim_{x \rightarrow \infty} \frac{3x^3 + 2x^2 - 5}{5x + 9x^3} = \frac{\infty}{\infty} \text{ ind.} = \frac{3}{9} = \boxed{\frac{1}{3}}$

② $P(x)$ -ren maila < $Q(x)$ -ren maila $\Rightarrow \lim_{\substack{x \rightarrow \infty \\ x \rightarrow -\infty}} \frac{P(x)}{Q(x)} = \boxed{0}$

Ad. $\lim_{x \rightarrow \infty} \frac{5x^2 + 2x}{-x^3 + 6} = \frac{\infty}{\infty} \text{ ind.} = 0$

$\lim_{x \rightarrow \infty} \frac{5x^2 + 2x}{-x^3 + 6} = 0$

③ $P(x)$ -ren maila > $Q(x)$ -ren maila $\Rightarrow \lim_{x \rightarrow \infty} \frac{P(x)}{Q(x)} = \begin{cases} \infty \\ -\infty \end{cases}$

$x \rightarrow \infty$ Fijatu $P(x)$ -ren maila altuen koefizientearen zeinua $x \rightarrow \infty$
 eta $Q(x)$ -ren maila altuen " zeinua $\frac{(+)}{(+)} \rightarrow (+)$; $\frac{(-)}{(+)} \rightarrow (-)$
 $x \rightarrow -\infty$ jaten bada antzekatzen gaitu beraketa eta $\frac{(+)}{(-)} \rightarrow (-)$; $\frac{(-)}{(-)} \rightarrow (+)$
 koefizientearen zeinua beraketa. $\lim_{x \rightarrow \infty} \frac{-3x^3 + 7x}{2x^4 + 5} = \frac{-\infty}{\infty} = -\infty$

$$\textcircled{8} \lim_{x \rightarrow 0} \frac{(1+x)^2 - 1}{x} = \frac{0}{0} \text{ indet} \quad \lim_{x \rightarrow 0} \frac{1 + x^2 + 2x - 1}{x} = \lim_{x \rightarrow 0} \frac{x^2 + 2x}{x} = \lim_{x \rightarrow 0} \frac{x(x+2)}{x} = \lim_{x \rightarrow 0} x+2 = 0+2 = \boxed{2}$$

$$\textcircled{9} \lim_{x \rightarrow 4} \frac{x^2 - 6x + 8}{x - 4} = \frac{0}{0} \text{ ind} \quad \lim_{x \rightarrow 4} \frac{(x-4)(x-2)}{(x-4)} = \lim_{x \rightarrow 4} x-2 = 4-2 = \boxed{2}$$

$$x^2 - 6x + 8 = (x-4)(x-2)$$

$$\begin{array}{r|rrr} 1 & -6 & 8 \\ 4 & & \\ \hline 1 & -2 & 0 \end{array}$$

$$\textcircled{10} \lim_{x \rightarrow 1} \frac{x^2 - 1}{x^3 - 1} = \frac{0}{0} \text{ ind} \quad \lim_{x \rightarrow 1} \frac{(x-1)(x+1)}{(x-1)(x^2+x+1)} = \lim_{x \rightarrow 1} \frac{x+1}{x^2+x+1} = \frac{1+1}{1^2+1+1} = \frac{2}{3}$$

$$x^2 - 1 = (x-1)(x+1)$$

$$\begin{array}{r|rrr} 1 & 0 & -1 \\ 1 & 1 & 1 \\ \hline 1 & 1 & 0 \end{array}$$

$$x^3 - 1 = (x-1)(x^2+x+1)$$

$$\begin{array}{r|rrrr} 1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 \end{array}$$

$$\textcircled{11} \lim_{x \rightarrow -1} \frac{x+1}{-1-x} = \frac{0}{0} \text{ ind} \quad \lim_{x \rightarrow -1} \frac{x+1}{-(x+1)} = \lim_{x \rightarrow -1} -1 = \boxed{-1}$$

$$\textcircled{12} \lim_{x \rightarrow 1} \frac{x^2 - 1}{\sqrt{x} - 1} = \frac{0}{0} \text{ ind} \quad \lim_{x \rightarrow 1} \frac{(x^2-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)} = \lim_{x \rightarrow 1} \frac{(x-1)(x+1)(\sqrt{x}+1)}{(x-1)} =$$

$$x^2 - 1 = (x-1)(x+1)$$

$$\lim_{x \rightarrow 1} (x+1)(\sqrt{x}+1) = (1+1)(\sqrt{1}+1) = 2 \cdot 2 = \boxed{4}$$

$$\textcircled{1} \lim_{x \rightarrow 1} \frac{x^3 + x^2 + 5}{x^3 - 2x^2 + x - 3} = \frac{1^3 + 1^2 + 5}{1^3 - 2 \cdot 1^2 + 1 - 3} = \frac{7}{-3} = \boxed{-\frac{7}{3}}$$

$$\textcircled{2} \lim_{x \rightarrow -2} \frac{x^3 + 5x^2 + 8x + 4}{x^3 + x^2 - 8x - 12} \stackrel{0/0 \text{ indet}}{=} \lim_{x \rightarrow -2} \frac{\cancel{(x+2)}(x+2)(x+1)}{\cancel{(x+2)}(x+2)(x-3)} = \lim_{x \rightarrow -2} \frac{x+1}{x-3} = \frac{-2+1}{-2-3} = \frac{-1}{-5} = \boxed{\frac{1}{5}}$$

$$\begin{array}{r|rrrr} x^3+5x^2+8x+4 & 1 & 5 & 8 & 4 \\ -2 & & -2 & -6 & -4 \\ \hline & 1 & 3 & 2 & 0 \\ -2 & & -2 & -2 & \\ \hline & 1 & 1 & 0 & \end{array}$$

$$(x+2)^2(x+1)$$

$$\begin{array}{r|rrrr} x^3+x^2-8x-12 & 1 & 1 & -8 & -12 \\ -2 & & -2 & 2 & 12 \\ \hline & 1 & -1 & -6 & 0 \\ -2 & & -2 & 6 & \\ \hline & 1 & -3 & 0 & \end{array}$$

$$(x+2)^2(x-3)$$

$$\textcircled{3} \lim_{x \rightarrow 3} \frac{x^3 - 4x^2 - 3x + 18}{x^3 + 3x^2 - 9x - 27} \stackrel{0/0 \text{ ind}}{=} \lim_{x \rightarrow 3} \frac{\cancel{(x-3)}(x-3)(x+2)}{\cancel{(x-3)}(x^2+6x+9)} = \lim_{x \rightarrow 3} \frac{(x-3)(x+2)}{x^2+6x+9} = \frac{(3-3)(3+2)}{3^2+6 \cdot 3+9} = \frac{0}{36} = \boxed{0}$$

$$x^3 - 4x^2 - 3x + 18 = (x-3)^2(x+2)$$

$$\begin{array}{r|rrrr} & 1 & -4 & -3 & 18 \\ 3 & & 3 & -3 & -18 \\ \hline & 1 & -1 & -6 & 0 \\ 3 & & 3 & 6 & \\ \hline & 1 & 2 & 0 & \end{array}$$

$$x^3 + 3x^2 - 9x - 27 = (x-3)(x^2+6x+9)$$

$$\begin{array}{r|rrrr} & 1 & 3 & -9 & -27 \\ 3 & & 3 & 18 & 27 \\ \hline & 1 & 6 & 9 & 0 \end{array}$$

$$\textcircled{4} \lim_{x \rightarrow 1} \frac{x-1}{\sqrt{x}-1} \stackrel{0/0 \text{ ind}}{=} \lim_{x \rightarrow 1} \frac{(x-1) \cdot \sqrt{x}-1}{\sqrt{x}-1 \cdot \sqrt{x}-1} = \lim_{x \rightarrow 1} \frac{\cancel{(x-1)}(\sqrt{x}-1)}{\cancel{(x-1)}} = \sqrt{1-1} = \boxed{0}$$

$$\textcircled{5} \lim_{x \rightarrow 3} \frac{x-3}{\sqrt{x}-\sqrt{3}} \stackrel{0/0 \text{ ind}}{=} \lim_{x \rightarrow 3} \frac{(x-3)(\sqrt{x}+\sqrt{3})}{(\sqrt{x}-\sqrt{3})(\sqrt{x}+\sqrt{3})} = \lim_{x \rightarrow 3} \frac{\cancel{(x-3)}(\sqrt{x}+\sqrt{3})}{\cancel{(x-3)}} = \sqrt{3}+\sqrt{3} = \boxed{2\sqrt{3}}$$

$$\textcircled{6} \lim_{x \rightarrow 1} \frac{x^3-1}{x^4-1} \stackrel{0/0 \text{ ind}}{=} \lim_{x \rightarrow 1} \frac{\cancel{(x-1)}(\sqrt{x})^2-(\sqrt{3})^2}{\cancel{(x-1)}(x^3+x^2+x+1)} = \lim_{x \rightarrow 1} \frac{x^2+x+1}{x^3+x^2+x+1} = \frac{1^2+1+1}{1^3+1^2+1+1} = \boxed{\frac{3}{4}}$$

$$\begin{array}{r|rrrr} & 1 & 0 & 0 & -1 \\ 1 & & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & 0 \end{array}$$

$$x^3-1 = (x-1)(x^2+x+1)$$

$$\begin{array}{r|rrrrr} & 1 & 0 & 0 & 0 & -1 \\ 1 & & 1 & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & 1 & 0 \end{array}$$

$$x^4-1 = (x-1)(x^3+x^2+x+1)$$

$$\textcircled{7} \lim_{x \rightarrow 3} \frac{x^2-6x+9}{x-3} = \lim_{x \rightarrow 3} \frac{\cancel{(x-3)}(x-3)}{\cancel{(x-3)}} = 3-3 = \boxed{0}$$

$$x^2-6x+9 = (x-3)(x-3)$$

$$\frac{\infty}{\infty}$$

Ez da bati indeterminazioa

(Zenb)

(izen)

(2)

Funtzio osoan (zenbakizailan eta izendatzailean)

maila altuen duen x begiratu. limitea

x honen koefizientea zenbakizailan zati

x honen koefizientea izendatzailean izango

da (bietako batean (izendatzaile edo zenbakizaila)

ez bada gai hori agertzen $\Rightarrow$ koefizientea 0 izango

da

zenb. eta
izen. bati-
garaitan
emanak
dardenera

kontuz!

$x \rightarrow +\infty$
 $x \rightarrow -\infty$

$\frac{k}{0}$ gertatzen

OHARRA

$\rightarrow -\infty$ -ra gertatzen denean
eta $\frac{k}{0}$ gertatzen $-\infty$
balioaz aldatzen da.

Kasu honetan mekanizazioa
zailagoa da zenbakizailan
eta izendatzailean

$x \rightarrow \infty$

maila
begirat
erabak
beretik

$x \rightarrow -\infty$ aurrekoa eta beretik

Kasu honetan
logikak
erabili
hobeto baita

$\frac{(+)}{(-)} \rightarrow (-)$; $\frac{(+)}{(+)} \rightarrow (+)$; $\frac{(-)}{(-)} \rightarrow (+)$

Ad: $\lim_{x \rightarrow \infty} \frac{3x^2 + 4x - 5}{x^2 - 3x + 3} = \frac{\infty/\infty \text{ Ind}}{1} = \frac{3}{1}$

maile altuena $x^2 \Rightarrow \frac{\text{bete koefizientea zenbakizailan}}{\text{bete koefizientea izendatzailean}}$

$\lim_{x \rightarrow \infty} \frac{x^5 + x^4 - 3x^3}{5x^3 + 6x^2} = \frac{\infty/\infty \text{ Ind}}{0} = +\infty$

maile altuena x^5

Kasu $\frac{k}{0}$ ez da indeter-
minazioa

$k > 0 \Rightarrow \infty$
 $k < 0 \Rightarrow -\infty$

$\lim_{x \rightarrow \infty} \frac{2x^2 + 3}{5x^5 + 4x^3} = \frac{\infty/\infty \text{ Ind}}{5} = 0$

$\lim_{x \rightarrow -\infty} \frac{x^2 - 6x + 8}{2x^2 - 5} = \frac{\infty/\infty \text{ Ind}}{2} = \frac{1}{2}$

maile altuena x^2

$\lim_{x \rightarrow -\infty} \frac{x^2 - 6x + 1}{x - 3} = \frac{\infty/\infty \text{ Ind}}{1} = -\infty$

maile altuena $x^2 \Rightarrow \frac{1}{0} = \infty$
 $\frac{b_1}{b_2} \rightarrow \frac{b_1}{b_2} \Rightarrow -$

LIMITEAK KALKULATZEKO ESKEMA

$$\lim_{x \rightarrow a} f(x)$$

1

1. Limitea Kalkulatzeak $x=a$ ordezkatu behar da funtzioan, hau egitean 2 gauza gerta daitezke: 1- Limitea lorzea (zehatza)
2. Indeterminazioa lorzea. (Indeterminazioa ebazte beharko duzu.)

Ad. 1- $\lim_{x \rightarrow 3} \frac{2x-5}{x^2+1} = \frac{2 \cdot 3 - 5}{3^2 + 1} = \frac{1}{10}$

2- $\lim_{x \rightarrow 1} \frac{2x^3 - 3x^2 + 1}{x^3 - 3x + 2} = \frac{2 \cdot 1^3 - 3 \cdot 1^2 + 1}{1^3 - 3 \cdot 1 + 2} = \frac{0}{0} \Rightarrow$ Indeterminazioa

INDETERMINAZIOEN EBAZPENAK ($\frac{0}{0}$; $\frac{\infty}{\infty}$; $\infty - \infty$; $\frac{\infty}{0}$; 1^∞ ...)

EBAZTEKO MEKANIKA (Pausoak)

- 1) Polinomioen arteko zatiketaren ematen deuen. $\frac{0}{0}$ $\rightarrow$ Ruffiniren edo beste bideen bidez izendatzaile eta zenbakitzaileak faktORIZATU. Bietan $(x-a)$ faktorea agertuko zaigu bidez, sinplifikatu eta limitea ebazte.
- 2) Erroak dituzten zatiketaren gertatzen deuen. $\rightarrow$ Zenbakitzaile eta izendatzaileak biderkatu erroa duen konjugatuaz! Biderkatu $(a-b)(a+b) = a^2 - b^2$ gertatzen den espresioan sinplifikatu eta ebazte limitea.

1) $\lim_{x \rightarrow 1} \frac{2x^3 - 3x^2 + 1}{x^3 - 3x + 2} = \lim_{x \rightarrow 1} \frac{(x-1)(x-1)(2x+1)}{(x-1)(x-1)(x+2)} = \frac{2 \cdot 1 + 1}{1 + 2} = \frac{3}{3} = 1$

$$2x^3 - 3x^2 + 1 = (x-1)(x-1)(2x+1)$$

$$\begin{array}{r|rrrr} 2 & -3 & 0 & 1 \\ 1 & & 2 & -1 & -1 \\ \hline & 2 & -1 & -1 & 0 \\ 1 & & 2 & 1 & \\ \hline & 2 & 1 & 0 & \end{array}$$

Bidan Ruffini lehen maila 1701 arte

$$x^3 - 3x + 2 = (x-1)(x-1)(x+2)$$

$$\begin{array}{r|rrrr} 1 & 0 & -3 & 2 \\ 1 & & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & 0 \\ 1 & & 1 & 2 & \\ \hline & 1 & 2 & 0 & \end{array}$$

Bidan Ruffini lehen maila 1701 arte, baturan ean da gehiago sinplifikatu

2) $\lim_{x \rightarrow 5} \frac{x-5}{\sqrt{x}-\sqrt{5}} = \lim_{x \rightarrow 5} \frac{(x-5)(\sqrt{x}+\sqrt{5})}{(\sqrt{x}-\sqrt{5})(\sqrt{x}+\sqrt{5})} = \lim_{x \rightarrow 5} \frac{(x-5)(\sqrt{x}+\sqrt{5})}{(x)-(\sqrt{5})^2} = \frac{(5-5)(\sqrt{5}+\sqrt{5})}{25-5} = \frac{0}{20} = 0$

$(\sqrt{x}-\sqrt{5}) \rightarrow$ Konjugatua $(\sqrt{x}+\sqrt{5})$

$\frac{\infty}{\infty}$ indeterminazioa eta $\frac{k}{0}$ indeterminazioa. (ZUZENKETA)

$$\textcircled{1} \lim_{x \rightarrow \infty} \frac{(2x+7)(x+5)}{(x-3)(4x-3)} = \lim_{x \rightarrow \infty} \frac{2x^2+17x+35}{4x^2-15x+9} = \frac{\infty/\infty \text{ ind.}}{\frac{2}{4}} = \frac{1}{2}$$

Pauso hau ez da beharrezkoa indeterminazio ebazteko aurrikustenik badiugu Zein izango den zenbakizkoaren maila (eta haren koefiziente) eta izendatzailearen (eta haren koefiziente).

$$\textcircled{2} \lim_{x \rightarrow -\infty} \frac{\textcircled{1} (x+2)(x-7)}{\textcircled{2} (x+1)(x-3)} = \frac{\infty/\infty \text{ ind.}}{\frac{1}{1}} = 1$$

① $x^2 + \dots$

② $x^2 + \dots$

$$\textcircled{4} \lim_{x \rightarrow \infty} \frac{(1+x)^2 - 1}{x} = \lim_{x \rightarrow \infty} \frac{x^2 + 2x - 1}{x} = \lim_{x \rightarrow \infty} \frac{\infty/\infty \text{ ind.}}{x} = \infty$$

$$\textcircled{5} \lim_{x \rightarrow \infty} \frac{x^2 - 6x + 1}{x - 3} = -\infty$$

$x \rightarrow \infty \quad \frac{1}{0} \rightarrow \infty$

$x \rightarrow -\infty \rightarrow \frac{\text{par}}{\text{impar}} \rightarrow \text{impar.} \quad \frac{(-\infty)^2}{-\infty} = -\infty$

$$\textcircled{3} \lim_{x \rightarrow \infty} \frac{x^3 - 1}{x^4} = \frac{\infty/\infty \text{ ind.}}{\frac{0}{1}} = 0$$

$$\textcircled{6} \lim_{x \rightarrow \infty} \frac{3x^2 - 4x + 7}{(x+2)^2} = \frac{\infty/\infty \text{ ind.}}{\frac{3}{1}} = 3$$

$x^2 + \dots$

$$\textcircled{7} \lim_{x \rightarrow \infty} \frac{2x+7}{\sqrt{4x^2+2x} + 3x} = \frac{\infty/\infty \text{ ind.}}{\frac{2}{\sqrt{4} + 3}} = \frac{2}{5}$$

$$\textcircled{8} \lim_{x \rightarrow \infty} x^2 + 5x - 14 \quad \frac{36}{0} \rightarrow \frac{k}{0} \text{ indet.}$$

$$\lim_{x \rightarrow 5^-} \frac{x^2 + 5x - 14}{x^2 - 10x + 25} = \infty$$

$$\textcircled{9} \lim_{x \rightarrow 3} \frac{2x}{x^2-9} = \frac{\frac{6}{0}}{0} \rightarrow \boxed{\text{ind } \frac{k}{0}}$$

$$\Leftrightarrow \begin{cases} \lim_{x \rightarrow 3^-} \frac{2x}{x^2-9} = -\infty \\ \lim_{x \rightarrow 3^+} \frac{2x}{x^2-9} = \infty \end{cases}$$

$$\textcircled{10} \lim_{x \rightarrow 3} \left(\frac{x^3-x^2-6x}{x^3-4x^2-3x+18} \right) = \frac{0}{0} \text{ ind.} = \lim_{x \rightarrow 3} \frac{x(x-3)(x+2)}{(x-3)(x-3)(x+2)} = \lim_{x \rightarrow 3} \frac{x}{x-3} = \frac{\frac{3}{0}}{0} \rightarrow \boxed{\text{ind } \frac{k}{0}}$$

$$\Leftrightarrow \begin{cases} \lim_{x \rightarrow 3^-} \frac{x}{x-3} = -\infty \\ \lim_{x \rightarrow 3^+} \frac{x}{x-3} = \infty \end{cases}$$

$$x^3-x^2-6x = x(x^2-x-6) = x(x-3)(x+2)$$

$$\begin{array}{r|rrrr} 3 & 1 & -1 & -6 & \\ & & 3 & 6 & \\ \hline & 1 & 2 & 0 & \end{array}$$

$$x^3-4x^2-3x+18 =$$

$$\begin{array}{r|rrrr} 3 & 1 & -4 & -3 & 18 \\ & & 3 & -3 & -18 \\ \hline & 1 & -1 & -6 & 0 \\ 3 & & 3 & 6 & \\ \hline & 1 & 2 & 0 & \end{array}$$

$$\textcircled{11} \lim_{x \rightarrow 2} \frac{6x-5}{4x^2-16x+16} = \frac{\frac{7}{0}}{0} \text{ ind } \frac{k}{0} = \boxed{\infty}$$

$$\begin{cases} \lim_{x \rightarrow 2^-} \frac{6x-5}{4x^2-16x+16} = \boxed{\infty} \\ \lim_{x \rightarrow 2^+} \frac{6x-5}{4x^2-16x+16} = \boxed{\infty} \end{cases}$$

$$\textcircled{12} \lim_{x \rightarrow \infty} \frac{3x + \sqrt{4x^2+3x-5}}{5x-3} = \frac{\infty}{\infty} \text{ ind.} = \frac{3+\sqrt{4}}{5} = \boxed{1}$$